MATHEMATICAL MODELING OF PLANT, PATHOGEN AND HERBIVORE INTERACTION INCORPORATING ALLEE EFFECT AND HARVESTING

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other University

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ABSTRACT

In ecology, interactions between various species that live in a similar habitat are common. In the plant-pathogen-herbivore interactions, plants are invaded by pathogens and herbivores while the herbivores are harvested by natural enemies such as predators and human. Harvesting the species may affect the population densities of the harvested species and keeping harvested species is critical in the ecosystem. On the other hand, the abundance of food does not guarantee exponential growth of species who reproduce sexually and species governed by carrying capacity. Therefore, the Allee effect may be crucial for sustaining such species. Recent plant-pathogen-herbivore models have not taken the Allee effect and species harvesting into account. The main objective of this study was to formulate and analyze a mathematical model of plant-pathogen-herbivore interactions incorporating Allee effect and harvesting. To illustrate the interaction of the species, the model was formulated using a system of ordinary differential equations. The local stability analysis was investigated and numerical simulations were performed using MATLAB software. The stability analysis showed that the ratio of intrinsic growth rate to the environmental carrying capacity of susceptible plants must be greater than certain threshold value to raise sufficient plant biomass to sustain other species. It also shows that the intrinsic growth rate of plants must be greater than the harvesting rate of plant population for plants to get established. Given this circumstance, all species coexists. Numerical simulations show that all species coexist when intrinsic growth rate of plants is greater than the harvesting rate and when conversion rate of what is eaten by herbivores to newborn ones is greater than that of their natural enemies. It also shows that in the absence of susceptible plants, herbivores migrates in search of food, while others deteriorate and die out. Furthermore, regardless of the availability of susceptible plants, the herbivores population goes to extinction if the herbivore population is less than the least number required to keep the herbivores existing in the ecosystem. In the interest of conservation of all species and the environment, policy developers will greatly benefit from understanding the solutions to address human activities for example, clearing land for farming, settlement, infrastructure construction, burning charcoal, and herbivore or their natural enemies hunting. In addition, monitor species closely, especially those that reproduce sexually by establishing and maintaining the least number required to keep the species existing.

DEDICATION

To my family and friends who never got tired of praying for my success and encouragement towards this thesis. Their patience and encouragement in every occasion is a great source of motivation.

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CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Ecology is the study of how different species including humans relate to their surroundings [25, 19]. The diverse behaviors displayed by these species in the ecosystem has sparked great interest in the formulation of dynamical models to illustrate the interaction of the species and their environment [7]. The concept of ecosystem was introduced by Jones et al. [16] who defined ecosystem engineers as species that generate, significantly modify, maintain, or destroy the resources (other than themselves) that are available to other organisms by generating physical change in living and non-living elements.

All types of organisms coexist in different habitats. Given that all species feed on their varied food sources to provide them with energy for life, growth, and development, their sources of food link them together within these ecosystems. For instance, plants manufacture their own food from water and sunlight, whereas animals feed on other species in order to survive. Therefore, the law of nature for all living things in every ecosystem is based on the struggle for food, with the weak being eliminated from the ecosystem while the strong species survive [4].

Interactions between species have an impact on an ecosystem biomass, productivity, and population size of each species. For instance, if there are two species, they must interact with each other because no species can survive on its own and all species depend on one another for survival. This interaction can be direct or indirect; for instance, herbivores consume plants, and predators consume herbivores, all of which have an impact on the ecosystem as a whole. As a result, the existence of all species is necessary for an ecosystem to be in balance. According to [3, 5], the entire ecosystem equilibrium will change if one type is abundant or scarce. Therefore, mathematical modeling in ecology takes into account the interaction of species and their habitat. For example, predator-prey interaction, plant-herbivore interactions, and interactions between herbivores and plant pathogens among others [3, 4].

The biological process of herbivory involves a species (herbivore) feeding on plants or their byproducts. This is one of the fundamental interactions between species in an ecosystem that shapes the natural habitats found in all ecosystems. The plant-herbivore interactions is situated between the primary production of plants and higher trophic levels at the base of food webs [11, 18]. This makes it easier for energy from plants to flow from consumers, predators, to decomposers [10, 15]. Therefore, plant-herbivore interactions may have an impact on ecosystem characteristics such as primary productivity and diversity of food webs among others [10].

According to [9], the presence of herbivores hinders the growth, development, and reproduction of plants. However, plants have developed many defense mechanisms against herbivores, which are divided into resistance and tolerance [1]. However, herbivores also assist plants in pollination and they add nutrients in the soil for plants. The most general model describing effects of plant growth and herbivore consumption on plant biomass is given by Rees and Brown [28] as given below

$$\frac{dB}{dt} = B[r(t) - n(t) - m(t)] \tag{1.1}$$

The description of the model parameters are in [28] where B is the plant biomass. This show that herbivores and environmental conditions contribute to the loss of plant biomass.

Pathogens also invades plants, according to [17]. Pathogens causes modifications that modify plant-insects interactions, growth and development. In turn, this modification has an impact on the kind and number of pollinators that the plants may attract, which may lower plant reproduction. On the other hand, pathogens may have an impact on plant-herbivore interactions by modifying plant qualities that serve as cues for herbivores [14]. This entails modifying the appearance and nutritional value which influence both herbivores preference and performance [20]. As a result, pathogens provide a serious threat to the survival of both plants and other species, such as herbivores.

Due to repeated invasions, plants have developed defense mechanisms that enable them to recognize herbivores or pathogens [22]. These mechanisms include the release of volatile organic compounds (VOC), which attract natural enemies of herbivore to lessen enemy pressure[27]. In addition, when they are attacked by herbivores, some plants generate herbivore-induced plant volatile(HIVP) which may reduce pathogens pressure [13].

Plants and other species have been harvested and mined from the ecosystem. Harvesting involves elimination of the species from the ecosystem [25]. For instance, through forest fire, prolonged drought, deforestation where plants are cleared for farming, settlement and charcoal burning. On the other hand, herbivores can be harvested through predators who prey on herbivores, diseases which may lead to death, migration and natural calamities like fire and drought. In addition, herbivores can be removed from the ecosystem through human activities on the system such as hunting of herbivores.

Harvesting species may cause genetic changes, for instance, alteration of population subdivision, loss of genetic variation among others. If harvesting is excessive, the population densities of harvested species may decrease and eventually be wiped out [25]. Therefore, the problem of sustaining the productivity of the populations being harvested is crucially important for ecosystem balance. The unpredicted collapse of many harvested species is just one illustration of the need to bring Allee effect to the forefront of conservation and management strategies of the ecosystem.

Warden Cyle Allee first described the Allee effect phenomena in 1931 [2]. Generally, the Allee effect is a fundamental ecological mechanism that establishes lower limits on a species density below which population goes to extinction. It outlines a favorable association between a species population and per-capita growth [12]. Allee effects have been observed empirically in a variety of organisms, including mammals, birds, plants, insects, and marine invertebrates. There are a variety of mechanism that create Allee effect, including mating systems, predation, environmental modification, the smallest group size essential to successfully rear offspring, produce seeds, increase genetic inbreeding [21, 29].

Therefore, plant-pathogen-herbivore system describes the interaction where plants serve as food for pathogen and herbivores. In this study, some plants are harvested (cleared for human activities). On the other hand, herbivores are harvested through hunting by their natural enemies or migration from one habitat to another. Moreover,the abundance of food does not guarantee exponential growth of species like herbivores who are assumed to reproduce sexually and species governed by the carrying capacity. Therefore, in this study, it is essential to investigate how Allee effect may be crucial for sustaining herbivores and how harvesting species may affect the ecosystem.

1.2 Statement of the Problem

In the ecosystem, plants interact in complex ways with herbivores and pathogens. Pathogenic-microbes, herbivores and harvesting of plants are major threats to plants survival. Plants are invaded by pathogen and herbivores while the herbivores are harvested by natural enemies such as predators and human. Harvesting of a species may affect the population densities of the harvested species. On the other hand the abundance of food does not guarantee exponential growth of species who reproduce sexually and the species governed by the carrying capacity. Therefore, understanding the dynamics of plants, pathogen, and herbivores may be important to the conservation of the species that are prone to extinction. Recent studies, for instance [23, 24], have been done on plant-pathogen-herbivore interactions. However, the models have not incorporated Allee effect and harvesting of species which are realities that govern the ecosystem behavior. Therefore, it is worth to investigate the impact of Allee effect and harvesting of species on the dynamics of plants-pathogens-herbivores interactions to enhance non-extinction of species.

1.3 Objective of the Study

1.3.1 Main Objective

The main objective of this study is to formulate and analyze a mathematical model for plant-pathogen-herbivore interactions incorporating harvesting and Allee effect.

1.3.2 Specific Objective

The specific objectives of the study are as follows:

(i) To formulate a mathematical model for plant-pathogen-herbivores interactions incorporating harvesting and Allee effect based on the system of ordinary differential equation.

(ii) To perform stability analysis of the model formulated by analyzing jacobian matrix around the equilibrium points.

(iii) To perform numerical simulations of the model developed using secondary data obtained from different literature so as to graphically illustrate the theoretical results obtained in objective (*ii*).

1.4 Justification of the Study

The study is motivated by the finding of [24] which shows that the ratio of intrinsic growth rate of plant and carrying capacity of the environment must exceed a certain threshold value for coexistence of all populations. Most species are prone extinction and harvesting of species may affects the species density. The plant population is sensitive to the feeding rate of herbivore and pathogen invasion. On the other hand, mortality rate of herbivores and pathogens depend on availability of food, that is, plants supply. It is worth to investigate how Allee effect and harvesting of species would affect dynamics of plant-pathogen-herbivore interaction model.

1.5 Significance of the Study

For the policy-makers and ecologists interested in ecosystem protection, a mathematical model that describes how plants, pathogens, and herbivores interact while taking into account the Allee effect and species harvesting is significant. Since model analysis provides insights into the variables that affect the stability and coexistence of the species, making it possible to maintain species and the ecosystem. Given the fact that any activity that threatens to upset the system balance may result in significant changes to the species densities and the ecosystem as a whole. Ecologists may also use the finding of this study to better understand and forecast the actions of plants, pathogens, herbivores and the natural enemies of herbivores in order to save the threatened species and the ecosystem.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Mathematical modeling has been a great tool for understanding species interaction dynamics which allows the policymakers to obtain useful biological insight and formulate the policies to maintain diversity of nature. This chapter highlight previous studies conducted considering different mathematical models in ecology.

2.2 Mathematical Modeling in Ecology

Basically, mathematical modeling in ecology takes into account how different species and the environment interact. These include, predator-prey interactions, plant-herbivore interactions, plant-pollinators interactions, plant-pathogen-herbivore interactions, competitive interactions, evolution of pesticides-resistance species among others. In this study, previous studies on the plant-herbivore interaction and plant-pathogen-herbivores interactions are taken into account.

2.3 Plant-Herbivore Models

Bandyopadhyay and Saha [7] formulated a plant-herbivore model. The model consists of plant and herbivore population denoted by $N(t)$ and $P(t)$ at any time t respectively. The two dimensional plant-herbivore system is governed by the equations;

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - \frac{aN^2P}{b + N^2}$$

$$\frac{dP}{dt} = \frac{eaN^2P}{b+N^2} - mP \tag{2.1}$$

The description of the model parameters are represented in [7]. The stability analysis of the model showed that the presence of a locally stable positive inner equilibrium point implies the coexistence of both species.

In the analysis of system of equation (2.1), the authors [7], assumed that plants are only attacked by herbivores with analogy of classical predator prey interaction and the amount of plant biomass destroyed follows Holling type-III functional response. However, this may not be the case in real life situation since plants are also affected by human activities where plants are cleared for farming, settlement and construction of infrastructures through natural habitat. Furthermore, the model did not incorporate other species and phenomenon including plant pathogens, Allee effect and natural enemies of herbivores whose behaviors govern the ecosystem.

Audrey *et al.* [6] formulated a mathematical model that accounts for both direct and apparent compensation in interactions between plants and herbivores. The model included two coupled ODEs that described the temporal evolution of plant biomass B and the population of herbivores H. The model is given by;

$$\begin{aligned} \frac{dB}{dt} &= r_B(H)B - \sigma(B)B - \phi(B)BH \\ \frac{dH}{dt} &= \alpha(B)H - \mu H \end{aligned} \tag{2.2}$$

The description of the model parameters are represented in Audrey *et al.* [6]. The stability analysis of the model were investigated. The results shows that if the initial population of herbivores is high, it may cause a substantial negative reaction in the plants, which may not sustain the population of herbivores over the long term and result in the extinction of plants and herbivores. However, herbivores become extinct due to famine if the initial density of herbivores is too low.

In the analysis of the system of equation (2.2), Audrey *et al.* [6] considered only herbivory. However, herbivores are harvested by predators and plants are harvested through human activities and are also invaded by pathogens. In addition, most species are prone to extinction especially those that are reproduced sexually, therefore, Allee effect should be forefront environmental management strategy.

Vijayalakshmi and Gunasekaran, [29] formulated a model for plant-herbivore system incorporating dynamics behavior of disease spread and Allee effect. The model was formulated by dividing the total population of species into three compartment, namely susceptible $x(t)$, infected $y(t)$ and herbivore population $z(t)$. The model is governed by the following system of the ODEs.

$$\begin{aligned} \frac{dx}{dt} &= rx(x-\theta)(1-x-y) - F(N)\frac{y}{N}x - H(x,N)z \\ \frac{dy}{dt} &= F(N)\frac{y}{N}x - H(y,N)z - \mu y \\ \frac{dz}{dt} &= z[mH(x,N) + \omega H(y,N) - \sigma \end{aligned} \tag{2.3}$$

The description of the model parameters are represented in [29]. In this study, stability of different equilibrium points were examined. The results revealed that species are prone to extinction and their initial population plays an important role in the survival of the species. In addition, numerical simulation was done to observe effects of diseases and Allee effect on the species density which indicates that susceptible plants and infected plants with Allee effect when $\theta = 0.3$, and $\mu = 1$ goes to extinction while susceptible plant and herbivores with Allee effect coexists when $\theta = 0.3$, and $\sigma = 0.6$. Where θ is the Allee threshold, μ is the death rate of infected plants and σ is the natural death rate of herbivores

In the analysis of the system of equation (2.3) Vijayalakshmi and Gunasekaran, [29] assumed that herbivores capture susceptible and infected plant at the same rate but the consumption of infected plant has less benefits or even cause harm to the herbivore. However, in real life situation, the herbivores cannot capture both susceptible and infected plants at the same rate. Herbivores are always sensitive on what they consume and must match their preference. Furthermore, the model has not incorporate harvesting and natural enemies of herbivores which influences the behavior of an ecosystem.

Asfaw *et al.* [3] formulated a model of plant-herbivore interaction with Allee effect. The model was governed by the classical predator-prey ecological model where the Allee effect for herbivores are taken into account. The model consists of plant population P and herbivore population H at any time t. The model was governed a system of nonlinear ordinary differential equations given below;

$$\begin{aligned} \frac{dP}{dt} &= P[r(1-\frac{P}{K})-HF(P)] \\ \frac{dH}{dt} &= H[cF(P)(\frac{H}{h+H})-D(F(P)] \end{aligned} \tag{2.4}$$

The description of the model parameters are represented in Asfaw *et al.* [4]. Analysis of the model dynamical behavior of the equilibrium points and stability of those equilibrium points was done. The study revealed that the plant population grows logistically bounded by the available resources and if the initial herbivore population is lower than the minimum required to maintain the population in the system, the herbivores are wiped out.

In the analysis of the system of equation (2.4), Asfaw *et al.* [3] considered classical predator-prey ecological model where Allee effect for herbivore was taken into account which may not be true since plants interacts with other species apart from herbivores. The model did not incorporate other environmental factors such as plant pathogens, harvesting of species and natural enemies of herbivores which in real life situation affect the interaction of plants and herbivores in the ecosystem. In addition, the numerical simulation was not performed to verify the theoretical results of the model.

Asfaw *et al.* [4] formulated a model for the co-existence thresholds in the dynamics of the plant-herbivore interaction with Allee effects and harvest. The model was governed by the classical predator-prey ecological model where the Allee effect for the herbivore is taken into account. The model consists of plant density P and herbivore density H. The model is governed by a system of nonlinear ordinary differential equations given below;

$$\begin{aligned} \frac{dP}{dt} &= rP(1-\frac{P}{K}) - HF(P) \\ \frac{dH}{dt} &= H[cF(P)(\frac{H}{h+H}) - DF(P) - \mu] \end{aligned} \tag{2.5}$$

The description of the model parameters are represented in Asfaw *et al.* [4]. The model dynamical behavior of equilibrium points and stability of those equilibrium points were analyzed which indicates that if the herbivore population is less than the least number of herbivore population required to keep the population existing then the herbivore population goes to extinction irrespective of other parameters. Furthermore, the numerical simulation of the model was performed. This revealed that, when $R_0 < 1$ and for any initial plant population $P(0) > 0$ the plant herbivores dynamics will always stabilize to the point $(K, 0)$. That is, the herbivore population will die out because the average number of newly born herbivores is less than one. If $R_0 > 1$ and choosing initial population from the appropriate interval, then all populations coexists. Where, R_0 is the reproduction number and K is the carrying capacity

In the analysis of the system of equation (2.5), Asfaw *et al.* [4] assumed the interaction of plants and herbivores is governed by classical predator-prey ecological model which in reality, may not be the case. The ecosystems is composed of many other species who depend on each other for survival. Furthermore,the model did not incorporate plant pathogens and natural enemies of herbivores which are common in ecosystem. In real life situation, The plants are attacked by pathogens which causes threat to plant survival while herbivores are attacked by their enemies who occupy the same ecosystem.

2.4 Plant-Pathogen-Herbivore Interactions

Mukherjee [23], examined a mathematical model of plant responses to diseases and herbivores where plants are subjected to disease. The model was formulated with four compartments at t where, $S(t)$ and $I(t)$ are the population of susceptible and infected plants respectively. $Y(t)$ and $Z(t)$ are the population density of herbivores and their natural enemies respectively. The model is governed by the following system of ordinary differential equation:

$$\begin{aligned}
\frac{dS}{dt} &= S[r(1-\frac{S}{K}) - \beta I - p_1 Y] \\
\frac{dI}{dt} &= I(\beta S - \mu) \\
\frac{dY}{dt} &= Y(-d_1 + c_1 p_1 S - P_2 Z) \\
\frac{dZ}{dt} &= -d_2 Z + c_2 p_2 Y Z + \omega
\end{aligned} \tag{2.6}$$

The description of the model parameters are represented in Mukherjee [23]. The analysis showed that the planar equilibrium point may be stable or unstable under certain restrictions on the system parameters, depending on the dynamical behavior of the model at various equilibrium points and the stability of those equilibrium points. The results shown that when the disease transmission rate surpasses a specific threshold value and the herbivore consumption rate remains below a specific value, the herbivore free equilibrium point becomes stable.

However, in the analysis of the system of equation (2.6) Mukherjee [23], did not incorporate environmental perturbation such as harvesting of species, Allee effect and plants self defense mechanisms which are common in the ecosystem. Harvest maybe be caused by human interference, fire, occurrence of drought. Allee effect was not taken into account yet its empirical evidence has been reported in many natural habitats including plants, mammals and insect where some species experience difficulties in finding the mates.

Mukherjee [24], analyzed a model of effects of constant immigration in plant-pathogen-herbivore interactions. The model was formulated with four compartments at time t, namely; $S(t)$, $I(t)$ be the number of susceptible and infected plant respectively. $Y(t)$ and $Z(t)$ are the population sizes of herbivores and their natural enemies respectively. The model is governed by:

$$\begin{aligned}
\frac{dS}{dt} &= S[r(1-\frac{S}{K}) - \frac{\beta I}{1+\alpha S(t)} - p_1 Y] \\
\frac{dI}{dt} &= I(\frac{\beta S(t)}{1+\alpha S(t)} - \omega) \\
\frac{dY}{dt} &= Y(-d_1 + c_1 p_1 S - P_2 Z) \\
\frac{dZ}{dt} &= -d_2 Z + c_2 p_2 Y Z + \mu
\end{aligned} \tag{2.7}$$

The description of the model parameters are represented in Mukherjee [24]. The stability conditions in terms of parameters, the dynamical behavior of the model at various equilibrium points and stability of those equilibrium points were investigated. The local stability of the model shows that the of ratio intrinsic growth rate of plants to carrying capacity must exceed a certain threshold in order for all species to coexist.

However, in the analysis of system of equation (2.7), Mukherjee [24], did not incorporate environmental perturbation such as harvesting of species and Allee effect which are common in the ecosystem. Harvesting may be caused by human interference where plant species are cleared for settlement, farming and construction of infrastructure through the natural habitat, fire and occurrence of drought. Allee effect was not taken into account yet unpredicted collapse of many harvested species is common in various habitat. This illustrates the need to bring Allee effect to the forefront of conservation and management strategies of the ecosystem.

2.5 Summary of the Models

Model	**Strength**	**Weakness/Gap**	**Ref**
Plant-herbivore model	The model was formulated with two compartments. That is, Plants and herbivores. The study shows that the existence of positive interior equilibrium point indicates the coexistence of both species	The model assumed that plants are only invaded by herbivores which is not the case in real life. Plants are affected by other environmental perturbation such as pathogens and harvesting while herbivores are affected by their natural enemies and Allee effect	[7]
Mathematical model incorporating direct and apparent compensation in plant-herbivore interaction	The model was formulated with two compartments. That is, the herbivores and plant biomass.	The model assumed other environmental perturbation and only herbivory was taken into account. For instance, the model did not incorporate pathogens, natural enemies of herbivore, Allee effect and harvesting of species	[5]

A Mathematical model of plant-herbivore system incorporating dynamics behavior of disease spread and Allee effect	The model was formulated with three compartments. That is, susceptible plants, infected plants and herbivore population.	The study assumed that herbivores captures susceptible and infected plants at same rate which may not be true. The model did not incorporate natural enemies of herbivores, harvesting of plant species and pathogens	[28]
Model of plant-herbivore interactions with Allee effect	The model was formulated with two compartments. That is, plant population and herbivore population	The model did not incorporate other environmental perturbation such as pathogens, natural enemies of herbivores and harvesting of species	[3]
Mathematical model for the coexistence thresholds in the dynamics of plant-herbivore interaction with Allee effect and harvest	The model was formulated with two compartments, that is, plant population and herbivore population.	The model had analog of classical predator-prey interactions and did not incorporate other environmental perturbation such as pathogens and natural enemies of herbivores	[4]
Mathematical model of plant response to disease and herbivores where plants are subjected to disease	The model was formulated with four compartments. That is, susceptible plants, infected plants, herbivores and natural enemies of herbivores	The model did not incorporate other environmental perturbation such as harvesting of species, Allee effect and self-defense mechanisms of plants	[19]
Effects of constant immigration in plant-pathogen-herbivore interaction	The model was formulated with four compartments at time (t). That is, susceptible plants, infected plants, herbivores and natural enemies of herbivores	The model did not incorporate harvesting of plant species where plants are cleared for settlement, farming, construction of infrastructure through natural habitats among others and Allee effect	[20]

Table 2.1 Summary of the Models

2.6 Research Gaps

In light of the aforementioned discussion, it is assumed that the interaction between plants and herbivores have an analogy of a classic predator-prey relationship. However, in real life situations, plants can be invaded by pathogens and human activities, which may result in the exploitation of plant population, whereas herbivores are invaded by predators. On the other hand, plant-pathogen-herbivore interactions incorporating enemies of herbivores assumed the environmental perturbation such as Allee effect and harvesting of species which are common in many ecosystems. Plant population have been cleared for human activities and settlement. Furthermore, many herbivores have been harvested by predators, human through hunting and killing the species for commercial purposes and some migrate from one habitat to another. In this study, Plant-Pathogen-Herbivore interactions is analyzed incorporating harvesting of species and Allee effect.

CHAPTER 3

MODEL FOMULATION AND MODEL ANALYSIS

3.1 Model Descriptions and Formulation

Mathematical modeling involves transformation of problems from real world phenomenon using mathematical ideas and language symbols into mathematically solvable equations and whose numerical and theoretical analysis are useful for describing and forecasting the original real life conditions without performing experiment [4, 25]. The model making procedure involves making assumptions, variables and parameters where the mathematical model generated defines physical states by a system of equation. In this study, a mathematical model for plant-pathogen-herbivore interaction incorporating Allee effect and harvesting is formulated using a system of ordinary differential equation.

3.1.1 The Model

In this study, different types of population densities at time t are considered. The plant population is divided into susceptible plant population denoted by $S(t)$ that comprises of the plant population that are at risk of being invaded by pathogens. The infected plant is denoted by $I(t)$ it comprises of the plant population that are already invaded by the pathogens. On the other hand, $H(t)$ and $Y(t)$ are the herbivore population and their natural enemies population respectively. In this study, parameters are introduced to represent Allee effect on herbivore equation and harvesting of species on susceptible, infected plant and herbivores population are harvested by their natural enemies. The model is governed by the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= S[r(1-\frac{S}{k}) - \frac{\eta I}{1+aS} - p_1 H - \epsilon] \\
\frac{dI}{dt} &= I[\frac{\eta S}{1+aS} - \sigma - \epsilon] \\
\frac{dH}{dt} &= H[c_1 p_1 S(\frac{H}{\theta + H}) - \mu - p_2 Y] \\
\frac{dY}{dt} &= Y[c_2 p_2 H - d] \qquad (3.1)
\end{aligned}$$

With initial conditions given by $S(0) > 0$, $I(0) > 0$, $H(0) > 0$ and $Y(0) > 0$

Where, r is the intrinsic growth rate of susceptible plants, k is the environmental carrying capacity, η is the pathogen transmission rate and a is the preventive measures taken by susceptible plants to protect themselves from invasion . The term p_1 and p_2 are the consumption rate of susceptible plant-herbivore and predation rate herbivore-natural enemies respectively. Furthermore, c_1 and c_2 are corresponding conversion rates of what is eaten to newborns by herbivores and natural enemies of herbivores respectively. The parameter, μ is the removal rate of herbivores in the habitat. The mortality rate of the natural enemies of herbivores is denoted by d and σ is the mortality rate of the infected plants due to pathogens attack. The parameter ϵ is the harvesting rate of susceptible and infected plant population while θ is the Allee threshold.

The assumption of this model are as follows:

(i) Herbivores feed on the susceptible plants thus the infected plants survive until killed by pathogens or harvested due to less attack by herbivores.

(ii) Infected plants are less attractive to pollinators than healthy plants thus no reproduction of infected plants.

(iii) Susceptible plant population grows bounded by the carrying capacity of the environment in absence of herbivores, pathogens and harvest.

(iv) The species interaction and consumption are assumed to be of the same type in any ecosystem. The only difference could be due to different kingdom or families which is typical for ecological systems.

The equation $\frac{dS}{dt} = S[r(1-\frac{S}{k})-\frac{\eta I}{1+aS}-p_1H-\epsilon]$ describes how susceptible plant populations are attacked by pathogens, herbivores, and harvesting of in the system of equation (3.1). The carrying capacity of the ecosystem controls the growth of the susceptible plant populations in the absence of pathogens, herbivores, and harvesting. Infected plant populations interact according to the equation $\frac{dI}{dt} = I[\frac{\eta S}{1+aS} - \sigma - \epsilon]$, where some plants die from pathogen invasion at a rate denoted by σ while others can be harvested at a rate denoted by ϵ. The rate of herbivore reproduction is expressed in the first term of the equation $\frac{dH}{dt} = H[c_1p_1S(\frac{H}{\theta+H}) - \mu - p_2Y]$. This shows that an individual herbivore will reproduce more if it eats more, and will cease to exist in the absence of susceptible plants, i.e. $c_1p_1(0)(\frac{H}{\theta+H}) = 0$.

Due to the fact that $c_1p_1S(\frac{H}{\theta+H})$ goes to zero when susceptible plant population density disappears. The coexistence of the herbivore population is very crucial and is protected by $(\frac{H}{\theta+H})$ because it is believed that herbivores reproduce sexually, each individual herbivore strives to locate mates or avoids inbreeding. Allee threshold takes care of this to ensure that the number of herbivores does not go to extinction.

Harvesting rate of plant population is denoted by ϵ, this may be caused by prolonged drought, floods, forest fires or human interference where plant population is cleared for settlement, farming and burning of charcoal. Therefore, the harvest rate of plants population may be unique or periodic causing both susceptible plant population and infected plant population to die out. On the other hand, the harvest rate, represented by μ, depicts the systemic actions that result in the removal of herbivores from the confined habitat. This may be caused by forest fires, prolonged droughts, human involvement, and migration.

3.2 Model Analysis

3.2.1 Invariant Region

It is crucial to demonstrate positivity and boundedness of the solutions of the system of equation (3.1) since the variables indicate biological population densities. Positivity denotes population survival, and boundedness denotes a growth limitation brought on by natural resource constraints. For the model to be mathematically and biologically well posed, the state variables $S(t)$, $I(t)$, $H(t)$ and $Y(t)$ at all time must be non-negative. This implies that the positive quadrant $\mathbb{R}_4^+ = [(S, I, H, Y) \in \mathbb{R}_4 : S > 0, I > 0, H > 0, Y > 0]$ is positively invariant. This will be done by showing positivity and boundedness of the formulated model. This is shown by the lemma as follows:

Lemma 3.2.1. *(Positivity) All solutions* $[S(t), I(t), H(t), Y(t)]$ *of the system of equation* (3.1) *starting in* $(S_0, I_0, H_0, Y_0) \in \mathbb{R}_4^+$ *remain positive for all* $t > 0$.

Proof. The positivity of $S(t), I(t), H(t), Y(t)$ can be verified by the equations:

$\frac{dS}{dt} = S[r(1 - \frac{S(t)}{k}) - \frac{\eta I}{1+aS(t)} - p_1 H(t) - \epsilon]$

Let $v = t$ then $dv = dt$. Substituting in equation above and integrating both sides, we have

$\frac{dS}{dv} = S[r(1 - \frac{S(v)}{k}) - \frac{\eta I}{1+aS(v)} - p_1 H(v) - \epsilon]$

$\frac{dS}{S} = [r(1 - \frac{S(v)}{k}) - \frac{\eta I}{1+aS(v)} - p_1 H(v) - \epsilon]dv$

$lnS = \int_0^t [r(1 - \frac{S(v)}{k}) - \frac{\eta I}{1+aS(v)} - p_1 H(v) - \epsilon]dv + S_0$

Introducing exponential, we have

$S(t) = S_0 exp \int_0^t [r(1 - \frac{S(v)}{k}) - \frac{\eta I}{1+aS(v)} - p_1 H(v) - \epsilon]dv$

Applying, the same on entire system of equation (3.1), we have

$\frac{dI}{dt} = I[\frac{\eta S(v)}{1+aS(v)} - \sigma - \epsilon]$

$I(t) = I_0 exp \int_0^t [\frac{\eta S(v)}{1+aS(v)} - \sigma - \epsilon]dv$

For,

$\frac{dH}{dt} = H[c_1 p_1 S(\frac{H}{\theta+H}) - \mu - p_2 Y]$.

$H(t) = H_0 exp \int_0^t [c_1 p_1 S(v)(\frac{H}{\theta+H}) - \mu - p_2 Y(v)]dv$

For

$\frac{dY}{dt} = Y[c_2 p_2 H(t) - d]$

$Y(t) = Y_0 exp \int_0^t [c_2 p_2 H(v) - d]dv$

with $S_0, I_0, H_0, Y_0 > 0$. If $S(0) = S_0 > 0$ then $S(t) > 0$ for all $t > 0$. The same argument is valid for $I(t)$, $H(t)$ and $Y(t)$. Hence $int(\mathbb{R}_4^+)$ is positively invariant set. □

Lemma 3.2.2. *(Boundedness) All solutions of system of equation* (3.1) *will lie in the region* $A = [(S, I, H, Y) \in \mathbb{R}_4^+ : 0 \leq S + I + H + Y \leq \frac{B}{\gamma}]$ *for all positive initial values* $(S(0), I(0), H(0), Y(0)) \in \mathbb{R}_4^+$
where $\gamma = \min(r, \sigma, \epsilon, \mu, d)$ *and* $B = rk + c_1 c_2$.

Proof. Let us consider the function $Z(t) = S + I + H + Y$

Taking the derivative along a solution of system of equation (3.1)

$\frac{dZ(t)}{dt} = S[r(1 - \frac{S}{k}) - \epsilon] - I(\sigma + \epsilon) - \mu H - dY$

For each $\gamma > 0$, the following inequality is satisfied:

$\frac{dZ}{dt} + \gamma Z \leq B + (\gamma - r)S + (\gamma - \epsilon)S + (\gamma - \sigma)I + (\gamma - \epsilon)I + (\gamma - \mu)H + (\gamma - d)Y$

Now choose γ such that $0 < \gamma = \min(r, \epsilon, \sigma, \mu, d)$ the the above equation can be written as

$\frac{dZ}{dt} + \gamma Z \leq B$

By comparison theorem [8], we obtain

$0 \leq Z(S(t), I(t), H(t), Y(t)) \leq \frac{B}{\gamma} + Z(S(0), I(0), H(0), Y(0))/e^{\gamma t}$

Taking limit when $t \to \infty$, we have

$0 \leq Z(S(t), I(t), H(t), Y(t)) \leq Lim_{t\to\infty} \frac{B}{\gamma} + Z(S(0), I(0), H(0), Y(0))/e^{\gamma t}$

$0 \leq Z(t) \leq \frac{B}{\gamma}$.

Hence the system of equation (3.1) is bounded. □

Clearly, the total population is bounded. Therefore, each sub-population S, I, H, Y is bounded for all future times. Thus the system of equation (3.1) is biologically and mathematically well posed.

3.2.2 Equilibrium Points

In order to find the equilibrium points or steady states of the model system, we set the right hand side of the system of equations (3.1) equal to zero. The following equilibrium points are clearly present in the system of equation (3.1):

$E_0 = (0, 0, 0, 0)$, $E_1 = (\frac{k(r-\epsilon)}{r}, 0, 0, 0)$,$E_2 = (0, 0, \frac{d}{c_2 p_2}, \frac{-\mu}{p_2})$, $E_3 = (S_3, I_3, 0, 0)$, $E_4 = (S_4, 0, H_4, Y_4)$, and the last equilibrium point of the system is $E_5 = (S_5, I_5, H_5, Y_5)$. Where:

$S_3 = \frac{-\epsilon-\sigma}{-\eta+a\epsilon+a\sigma}$

$I_3 = \frac{r-\epsilon-\frac{ra(-\epsilon-\sigma)^2}{k(-\eta+a\epsilon+a\sigma)^2}-\frac{r(-\epsilon-\sigma)}{k(-\eta+a\epsilon+a\sigma)}+\frac{ra(\epsilon-\sigma)}{(-\eta+a\epsilon+a\sigma)}-\frac{dp_1}{c_2p_2}-\frac{da(-\epsilon-\sigma)p_1}{(-\eta+a\epsilon+a\sigma)c_2p_2}}{\eta}$

$S_4 = \frac{-dkp_1+krc_1p_2-k\epsilon c_2p_2}{rc_2p_2}$

$H_4 = \frac{d}{c_2p_2}$

$Y_4 = \frac{-d^2kc_1p_1^2-dr\mu c_2p_2+dkrc_1c_2p_1p_2-dk\epsilon_1c_2p_1p_2-r\theta\mu c_2^2p_2^2}{rc_2p_2^2(d+\theta c_2p_2)}$

$S_5 = \frac{-\epsilon-\sigma}{-\eta+a\epsilon+a\sigma}$

$$I_5 = \frac{r-\epsilon-\frac{ra(-\epsilon-\sigma)^2}{k(-\eta+a\epsilon+a\sigma)^2}-\frac{r(-\epsilon-\sigma)}{k(-\eta+a\epsilon+a\sigma)}+\frac{ra(\epsilon-\sigma)}{(-\eta+a\epsilon+a\sigma)}-\frac{dp_1}{c_2p_2}-\frac{da(-\epsilon-\sigma)p_1}{(-\eta+a\epsilon+a\sigma)c_2p_2}}{\eta}$$

$$H_5 = \frac{d}{c_2p_2}$$

$$Y_5 = -\mu + \frac{(-\epsilon-\sigma)c_1p_1}{-\eta+a\epsilon+a\sigma} - \frac{\theta(-\epsilon-\sigma)c_1p_1}{-\eta+a\epsilon+a\sigma(\theta+\frac{d}{c_2p_2})}$$

3.2.3 Local Stability

Stability analysis examines the solutions of differential equation formulated and trajectories of dynamical systems under small perturbations of initial conditions. In this study, local stability analysis of the system of equation (3.1) is performed. This involves examining the jacobian matrix of the model around the equilibrium points where the characteristic roots(eigenvalues) from characteristic equations are obtained. Using these eigenvalues, the behavior of the solutions of the model can be analyzed.

To examine the local stability of the equilibrium points E_0, E_1, E_2,E_3, E_4 and E_5, the eigenvalues of the jacobian matrix of the system of equation (3.1) around the equilibrium points is determined. The jacobian matrix of the system of equation (3.1) at any given point $J(S, I, H, Y)$ is given by:

$$J(E) = \begin{bmatrix} b_{11} & \frac{\eta S}{(1+aS)} & -p_1S & 0 \\ \frac{\eta I}{(1+aS)^2} & \frac{\eta S}{1+aS} - \sigma - \epsilon & 0 & 0 \\ Hc_1p_1(\frac{H}{(\theta+H)}) & 0 & c_1p_1S(\frac{H(2\theta+H)}{(\theta+H)^2}) - \mu - p_2Y & -p_2H \\ 0 & 0 & c_2p_2Y & c_2p_2H - d \end{bmatrix} \quad (3.2)$$

Where, $b_{11}=r[1-\frac{2S}{k}] - \frac{\eta I}{(1+aS)^2} - p_1H - \epsilon$

The stability of the equilibrium points are determined by the nature of the eigenvalues of the jacobian matrix evaluated at the corresponding equilibrium points. Evaluating the Jacobian matrix at the population free equilibrium point $E_0 = (0, 0, 0, 0)$ takes the form;

$$J(E_0) = \begin{bmatrix} r-\epsilon & 0 & 0 & 0 \\ 0 & -\sigma-\epsilon & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -d \end{bmatrix} \tag{3.3}$$

Where the eigenvalues of $J(E_0)$ are give by $\lambda_1 = -\mu$,$\lambda_2 = r-\epsilon$, $\lambda_3 = -\sigma-\epsilon$ and $\lambda_4 = -d$ which are real. Clearly, $E_0 = (0,0,0,0)$ is which is stable for $r < \epsilon$ and unstable for $r > \epsilon$. Therefore, regardless of the values of other parameters, the ecological species do not exist at the population-free equilibrium point. This could happen as a result from the occurrence of prolonged droughts or forest fires. These occurrences could result in the extinction of all species in the habitat. However, when $r > \epsilon$, the plant population may regenerate.

At equilibrium point $E_1 = (\frac{k(r-\epsilon)}{r}, 0, 0, 0)$, the Jacobian matrix takes the form:

$$J(E_1) = \begin{bmatrix} -r+\epsilon & 0 & -p_1\frac{k(r-\epsilon)}{k} & 0 \\ 0 & -\sigma-\epsilon & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -d \end{bmatrix} \tag{3.4}$$

The eigenvalues of $J(E_1)$ are given by $\lambda_1 = -\mu$, $\lambda_2 = -r+\epsilon$, $\lambda_3 = -\sigma-\epsilon$, $\lambda_4 = -d$ which are real. Therefore E_1 is locally asymptotically stable for $r > \epsilon$ otherwise unstable if $r < \epsilon$. This shows that the population of susceptible plants can grow logistically up to the environmental carrying capacity in the absence of pathogens, herbivores, and a low rate of harvesting. This is a biological observation because, given a certain piece of land with sufficient soil resources, no pathogens or herbivores, and a low harvest rate, plant populations can grow to the maximum extent that the land would support. The absence of herbivores in a limited ecosystem also ensures the extinction of herbivores' natural enemies. The system is stable when intrinsic growth rate of susceptible plant is greater than their harvesting rate $(r > \epsilon)$ otherwise, the system is unstable when intrinsic growth rate is less than the harvesting rate $(r < \epsilon)$.

The Jacobian matrix evaluated at $E_2 = (0, 0, \frac{d}{c_2 p_2}, \frac{-\mu}{p_2})$ takes the form:

$$J(E_2) = \begin{bmatrix} r - \epsilon - p_1 H_2 & 0 & 0 & 0 \\ 0 & -\sigma - \epsilon & 0 & 0 \\ H_2 c_1 p_1 (\frac{H_2}{\theta + H_2}) & 0 & -\mu & \frac{-d}{c_2} \\ 0 & 0 & -\mu c_2 & 0 \end{bmatrix} \tag{3.5}$$

Where the eigenvalues of $J(E_2)$ are given by $\lambda_1 = r - \epsilon - p_1 \frac{d}{c_2 p_2}$, $\lambda_2 = -\sigma - \epsilon$, $\lambda_3 = \frac{1}{2}(-\mu - \sqrt{\mu(4d + \mu)})$ and $\lambda_4 = \frac{1}{2}(-\mu + \sqrt{\mu(4d + \mu)}$ which are real. The equilibrium point E_2 is a saddle point which is unstable for $r > \epsilon + p_1 \frac{d}{c_2 p_2}$ otherwise stable for $r < \epsilon + p_1 \frac{d}{c_2 p_2}$. Since no species exists in isolation and herbivore survival is fully dependent on the availability of food, in this case plants, the herbivore population will starve to death since the herbivores have little or no food available. Therefore, in the absence of populations of susceptible plants, the herbivore populations lacks a source of food and eventually become extinct. Furthermore, since the population of herbivores is declining as a result of a lack of food, the herbivores' natural enemies gradually die out or move to another area thus the system is unstable.

This situation is evident over the world, especially in arid and semi-arid regions and during extended droughts when plant populations decline owing to a lack of water or soil nutrients. For instance, Osborne [26] stated that during the drought of 1993 in Kenya, fewer herbivores were present in several ecosystems. According to report, 70 percent of them perished from famine as a result of a lack of a source of food. This highlights the necessity of caring for the plant population, which serves as food to herbivores, pathogens and a water catchment region. Evidently, lack of populations of plant species causes herbivores to die out, which causes the natural enemies of herbivores in the constrained habitat to become extinct.

Theorem 3.2.1. *If* $\frac{r}{k} > r - \frac{\eta I_3}{(1+aS_3)^2} - \epsilon$ *and* $\frac{\eta S_3}{1+aS_3} < \sigma + \epsilon$ *then* E_3 *is locally asymptotically stable.*

Proof. The jacobian matrix evaluated at $E_3 = (S_3, I_3, 0, 0)$ takes the form:

$$J(E_3) = \begin{bmatrix} r(1 - \frac{2S_3}{k}) - \frac{\eta I_3}{(1+aS_3)^2} - \epsilon & \frac{\eta S_3}{1+aS_3} & -p_1 S_3 & 0 \\ \frac{\eta I_3}{(1+aS_3)^2} & \frac{\eta S_3}{1+aS_3} - \sigma - \epsilon & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -d \end{bmatrix} \quad (3.6)$$

Clearly, the first two eigenvalues of $J(E_3)$ are given by $\lambda_1 = -d$ and $\lambda_2 = -\mu$. The other eigenvalues are given by the following characteristic equation;

$\lambda^2 - (A - E)\lambda + BC = 0$. Solving this equation using quadratic formula, we obtain

$\lambda_3 = \frac{1}{2}(A - E - \sqrt{A^2 - 2AE + E^2 + 4BC})$,

$\lambda_4 = \frac{1}{2}(A - E + \sqrt{A^2 - 2AE + E^2 + 4BC})$

Where $A = r(1 - \frac{2S_3}{k}) - \frac{\eta I_3}{(1+aS_3)^2} - \epsilon$, $B = \frac{\eta S_3}{1+\alpha S_3}$, $C = \frac{\eta I_3}{(1+aS_3)^2}$ and $E = \frac{\eta S_3}{1+aS_3} - \sigma - \epsilon$ □

According to the theorem, the other two eigenvalues from the equation $\lambda^2 - (A - E)\lambda + BC = 0$ are real and are negative when $\frac{r}{k} > r - \frac{\eta I_3}{(1+aS_3)^2} - \epsilon$ and $\frac{\eta S_3}{1+aS_3} < \sigma + \epsilon$. Therefore, the equilibrium point E_3 is locally asymptotically stable under certain restriction. Evidently, the susceptible plant populations grows logistically to a specific threshold value needed to maintain the population when the rate of pathogen transmission and harvesting of the susceptible population is less than the intrinsic growth rate of the susceptible plant population. This holds in the absence of herbivores, who depend on populations of susceptible plants to survive, and when there is less harvesting of plant population.

Theorem 3.2.2. *The equilibrium point* $E_4 = (S_4, 0, H_4, Y_4)$ *is locally asymptotically stable if* $\frac{r}{k} > r - \epsilon - p_1 H_4$ *and* $\frac{\eta S_4}{1+aS_4} < \sigma + \epsilon$

Proof. The Jacobian matrix evaluated at E_4 takes the form:

$$\begin{bmatrix} b_{12} & -\frac{\eta S_4}{1+aS_4} & -p_1 S_4 & 0 \\ 0 & -\sigma - \epsilon + \frac{\eta S_4}{1+aS_4} & 0 & 0 \\ H_4 c_1 p_1 (\frac{H_4}{\theta+H_4}) & 0 & -\mu - p_2 Y + c_1 p_1 S_4 (\frac{H_4(2\theta+H_4)}{(\theta+H_4)^2)} & 0 \\ 0 & 0 & c_2 p_2 Y_4 & -d + c_2 p_2 H_4 \end{bmatrix} \quad (3.7)$$

Where $b_{12} = r(1 - \frac{2S_4}{k}) - \epsilon - p_1 H_2$

Clearly, the first two eigenvalues at $J(E_4)$are given by $d > c_2 p_2 H_4$ and $\epsilon + \sigma > \frac{\eta S_4}{1+aS_4}$. The other two eigenvalues are given by the following characteristic equation.

$\lambda^2 - (A + F)\lambda - E p_1 S_4 = 0$. Using quadratic formula, we obtain

$\lambda_3 = \frac{1}{2}(A + F - \sqrt{A^2 - 2AF + F^2 - 4Ep_1 S_4})$

$\lambda_4 = \frac{1}{2}(A + F + \sqrt{A^2 - 2AF + F^2 - 4Ep_1 S_4})$

where $A = r(1 - \frac{2S_4}{k}) - \epsilon - p_1 H_2,\ F = -\mu - p_2 Y + c_1 p_1 S_4 (\frac{H_4(2\theta+H_4)}{(\theta+H_4)^2)}$ and $E = H_4 c_1 p_1 (\frac{H_4}{\theta+H_4})$ □

The theorem implies that all the four eigenvalues at $J(E_4)$ are real and have negative signs from the theorem conditions . Therefore, E_4 is locally asymptotically stable when $\frac{r}{k} > r - \epsilon - p_1 H_4$ and $\frac{\eta S}{1+aS} < \sigma + \epsilon$. The absence of infected plant population implies absence of pathogen in the system. Therefore, the plant population, herbivore population and natural enemies of herbivore population can coexist.

For the coexistence of the three species, the initial susceptible plant population must be greater than the minimum required to sustain the herbivore population. Similar to this, the initial population of herbivores should be greater than the required number to maintain the population and provide food for natural enemies. For herbivores to be ensured food availability, the average density of susceptible plant populations growth must be able to sustain each population

Theorem 3.2.3. *If* $\frac{r}{k} > r - \frac{\eta I_5}{(1+aS_5)^2} - \epsilon$, *then* E_5 *is locally asymptotically stable*

Proof. At positive interior equilibrium point of system of equation (3.1) about $E_5 = (S_5, I_5, H_5, Y_5)$ where

E_5 is feasible if

$$r - \epsilon - \frac{ra(-\epsilon-\sigma)^2}{k(-\eta+a\epsilon+a\sigma)^2} - \frac{r(-\epsilon-\sigma)}{k(-\eta+a\epsilon+a\sigma)} + \frac{ra(\epsilon-\sigma)}{(-\eta+a\epsilon+a\sigma)} - \frac{dp_1}{c_2p_2} - \frac{da(-\epsilon-\sigma)p_1}{(-\eta+a\epsilon+a\sigma)c_2p_2} > 0$$

The Jacobian matrix evaluated at $J(E_5)$ reduces to

$$\begin{bmatrix} b_{13} & -\frac{\eta S_5}{1+aS_5} & -p_1S_5 & 0 \\ \frac{\eta I_5}{(1+aS_5)^2} & -\sigma - \epsilon + \frac{\eta S_5}{1+aS_5} & 0 & 0 \\ H_5c_1p_1(\frac{H_5}{\theta+H_5}) & 0 & -\mu - p_2Y + c_1p_1S_5(\frac{H_5(2\theta+H_5)}{(\theta+H_5)^2}) & 0 \\ 0 & 0 & c_2p_2Y_5 & -d + c_2p_2H_5 \end{bmatrix} \tag{3.8}$$

Where $b_{13} = r - \frac{2rS_5}{k} - \frac{\eta I_5}{(1+aS_5)^2} - \epsilon - p_1H_5$

Choosing a positive definite function about E_5 given as

$W(t) = r_1(S - S_5)S(t) + r_2(I - I_5)I(t) + r_3(H - H_5)H(t) + r_4(Y - Y_5)Y(t)$

where r_1,r_2,r_3 and r_4 are positive constants chosen to be: $r_1 = 1$,$r_2 = 1+aS_5$, $r_3 = \frac{1}{c_1}$, and $r_4 = \frac{1}{c_1c_2}$

Differentiate W with respect to t along the solution of system of equation (3.1) we get

$\frac{dW}{dt} = r_1(S - S_5)\frac{dS(t)}{dt} + r_2(I - I_5)\frac{dI(t)}{dt} + r_3(H - H_5)\frac{dH(t)}{dt} + r_4(Y - Y_5)\frac{dY(t)}{dt}$

$\frac{dW}{dt} = r_1(S - S_5)[S(r(1 - \frac{S}{k}) - \frac{\eta I}{1+aS} - p_1H - \epsilon)] + r_2(I - I_5)[I(\frac{\eta S}{1+aS} - \sigma - \epsilon)] + r_3(H - H_5)[H(c_1p_1S(\frac{H}{\theta+H}) - \mu - p_2Y)] + r_4(Y - Y_5)[Y(c_2p_2H - d)]$

Expanding $\frac{dW}{dt}$ about E_5, we obtain

$\frac{dW}{dt} = r_1(S-S_5)^2[r-\frac{2rS_5}{k}-\frac{\eta I_5}{(1+aS_5)^2}-\epsilon]-r_1p_1(S-S_5)(H-H_5)+r_2[\frac{\eta}{(1+aS_5)^2}(I-I_5)(S-S_5)]-r_2(\sigma-\epsilon)(I-I_5)+r_3c_1p_1(\frac{H_5(2\theta+H_5)}{(\theta+H_5)^2)})(H-H_5)(S-S_5)-r_3p_2(H-H_5)(Y-Y_5)-r_3\mu(H-H_5)+r_4c_2p_2(H-H_5)(Y-Y_5)-r_4d(Y-Y_5)$+Higher Order Terms

Such that the cross product $(S-S_5)(H-H_5)$, $(H-H_5)(Y-Y_5)$ and $(I-I_5)(S-S_5)$ equals to zero and we obtain

$\frac{dW}{dt} = r_1(S-S_5)^2[r-\frac{r}{k}-\frac{\eta I_5}{(1+aS_5)^2}-\epsilon]-r_2(\sigma+\epsilon)(I-I_5)-r_3\mu(H-H_5)-r_4d(Y-Y_5)$

Hence if $\frac{r}{k} > r-\frac{\eta I_5}{(1+aS_5)^2}-\epsilon$ then $\frac{dW}{dt}$ is negative definite everywhere so that the value of W is decreasing along the solutions and W is a lyapunov function about E_5. The solution implies that on any level, they curve into the region bounded. Thus, E_5 is locally asymptotically stable. □

The existence of locally stable positive interior equilibrium ensures the coexistence of susceptible plant population, infected plant population, herbivore population and natural enemies of herbivores in the system. Therefore, the susceptible plant population grows to the carrying capacity when the intrinsic growth rate of plants r is greater that the rate of pathogen attack η and harvesting rate ϵ. Furthermore, herbivore population increases since $c_1 > 1$ as a result of availability of food. Similarly, natural enemies of herbivores increase since $c_2 > 0.1$. To maintain stability of the system, activities on the system that increase mortality rate of species should be controlled.

From biological point of view, the existence of E_5 demand

(i) The ratio of the intrinsic growth rate to carrying capacity for susceptible plant population must be greater than some threshold value to raise the plant biomass for herbivores and pathogens to feed on and become established.

(ii) The conversion rate of plant biomass eaten by herbivore to give rise to newborn must be greater than their harvest rate and predation rate of natural enemies to sustain the natural enemies population to guarantee the non-extinction of any species.

(iii) The harvesting rate of plant species must be less than the intrinsic growth rate of susceptible plants for plants to be established.

This ensures the long term survival and persistence of all population density, that is, none of the species goes to extinction.

CHAPTER 4

NUMERICAL SIMULATION OF THE MODEL

In this study, numerical simulations are performed by the use of MATLAB software using secondary data obtained from [4, 24]. These simulations are performed to analyze the effect of harvesting of species and Allee effect on the ecosystem where time is in years. This help to verify theoretical results obtained graphically. The results obtained will give more insights and prediction of long term behaviour of the solutions.

4.1 Description of Parameters

Parameter	Description	Units
r	Intrinsic growth rate of susceptible plants	Per year
k	Environmental carrying capacity	Assumed
η	Pathogen transmission rate	Per infected plant
a	Measure of inhibition effects taken by susceptible plants to protect themselves	per susceptible plants
p_1	Predation rate of plant-herbivore	per herbivore
p_2	Predation rate of herbivore-natural enemies	per natural enemy
c_1	Conversation rate of what is eaten to newborns by herbivores	per herbivore
c_2	Conversation rate of what is eaten to newborns by natural enemies of herbivore	per natural enemy of herbivores
ϵ	Harvest rate of plants	per total plant population
σ	Mortality rate of infected plants due to pathogens attack	per total infected plant population
θ	Allee threshold	least herbivore number per total herbivore population
μ	removal rate of herbivores in the confined habitat	per total herbivore population
d	mortality rate of the natural enemies of herbivores	per total natural enemy population

Table 4.1: Description of the model parameters

4.2 Simulation for Susceptible and Infected Plants Interactions

To illustrate theorem 3.2.1, simulation is performed over time in years using the values adopted from [24] and others assumed as summarized in the Table 4.2.

FIGURE	r	k	η	a	σ	ϵ
Figure 4.1	4.5	5000	2.5	2	1.5	0.25
Figure 4.2	4.5	5000	3.4	2	1.5	5
Figure 4.3	4.5	5000	5	2	1.5	0.05

Table 4.2: Parameters value of susceptible and infected plants incorporating harvesting The numerical simulation of susceptible and infected plants gives Figure 4.1, Figure 4.2 and Figure 4.3 as shown below

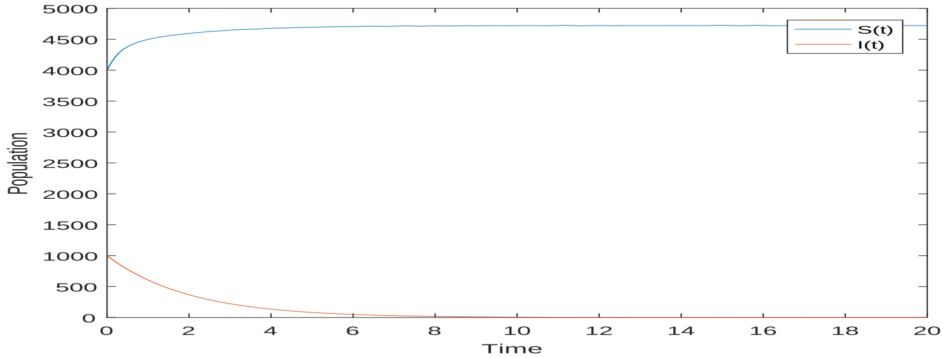

Figure 4.1: Susceptible and infected plants interaction with $\eta = 2.5$, $\epsilon = 0.25$

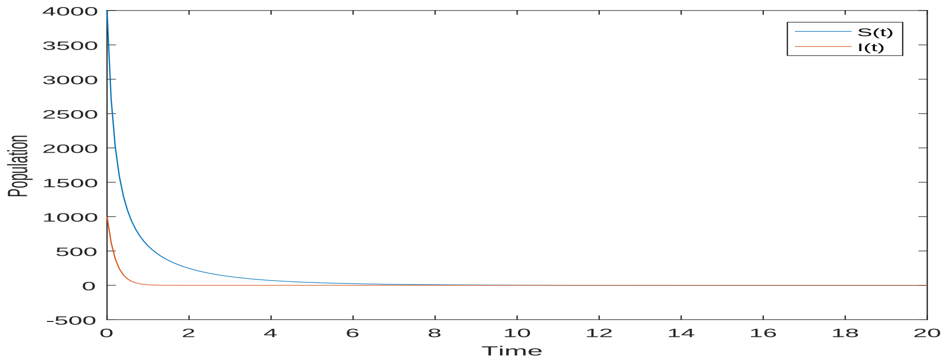

Figure 4.2: Susceptible and infected plants interaction with $\eta = 3$, $\epsilon = 5$

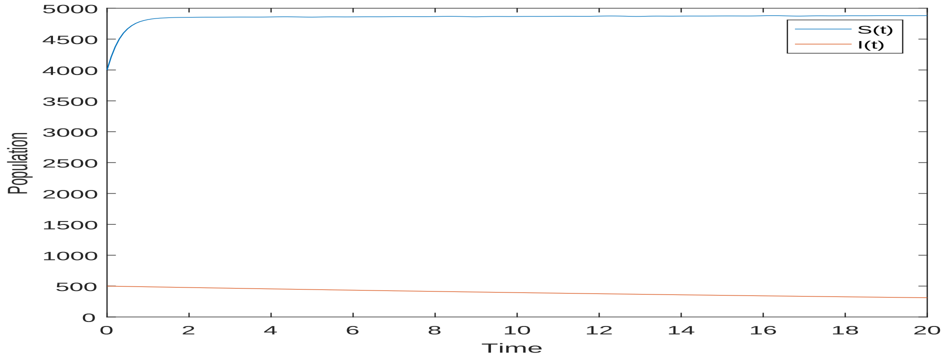

Figure 4.3: Susceptible and infected plants interaction with $\eta = 5$, $\epsilon = 0.05$

From figure 4.1 when the intrinsic growth rate of susceptible plant density is greater than the transmission rate of pathogen or the susceptible plants are resistant to pathogens, the infected plants density reduces and goes to extinction. Furthermore, susceptible plant density grows to the environmental carrying capacity when transmission rate of pathogen is $\eta = 2.5$ and harvesting rate of plant population is $\epsilon = 0.25$. On the other hand when $\eta = 3.4$ and $\epsilon = 5$, the susceptible plant density and infected plant density becomes extinct over a time as seen in figure 4.2. From theorem 1, under certain restriction, the susceptible plants density and infected plant density coexists as seen in figure 4.3. In addition, pathogens may survive based on the prevailing climatic conditions and some pathogens must be at a critical life stage for them to cause infections hence during this period, the susceptible and infected plants coexists.

4.3 Simulation of Susceptible Plants, Herbivores and their Natural Enemies Interaction

In the absence of pathogens, we have susceptible plants, herbivores and their natural enemies. Performing numerical simulation for this population dynamics using the data summarized in the Table 4.3 where the data is adopted from [4, 24], we get the graphs in Figure 4.4, Figure 4.5 and Figure

4.6

Figure	r	k	p_1	c_1	θ	μ	p_2	c_2	d	ϵ
Figure 4.4	15	10000	4	10	0.4	1	0.2	0.2	2	0.1
Figure 4.5	20	10000	0.4	1.22	0.4	-0.14	0.2	0.2	14	0.001
Figure 4.6	20	10000	0.002	1.22	0.4	-0.14	0.12	0.02	6	0.001

Table 4.3: Parameters value of susceptible plants, herbivores and their natural enemies interaction

The simulation of susceptible plants, herbivores and their natural enemies of herbivores are as follows:

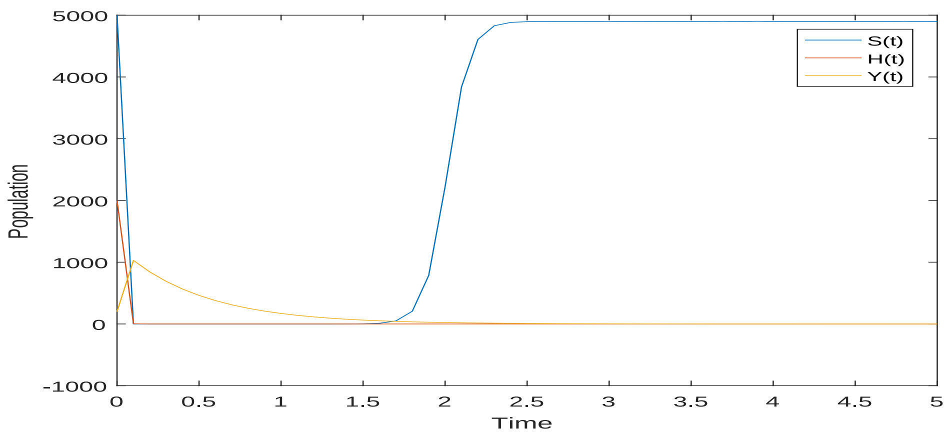

Figure 4.4: Susceptible plants, herbivores and natural enemies of herbivores interaction with $p_1 = 4$, $\epsilon = 0.1$

From figure 4.4, the susceptible plant population decreases drastically when the predation rate of herbivore and harvesting rate is high, that is, $p_1 = 4$ and $\epsilon = 0.1$. This leads to increase of natural enemies of herbivores population due to availability of food(herbivores). When susceptible plant population becomes extremely small, the herbivores decrease and goes to zero implying the herbivores dies out or some migrate away as they look for food hence extinction of herbivore at that confined habitat. Similarly, the natural enemies of herbivores also die out or migrate to different habitat looking for food due to decrease of herbivore. In long run, the susceptible plant population regenerates, grows and eventually reaches the carrying capacity of the environment. This attract the herbivores who in turn attract their natural enemies back in the same habitat and the cycle occurs again as seen in Figure 4.5

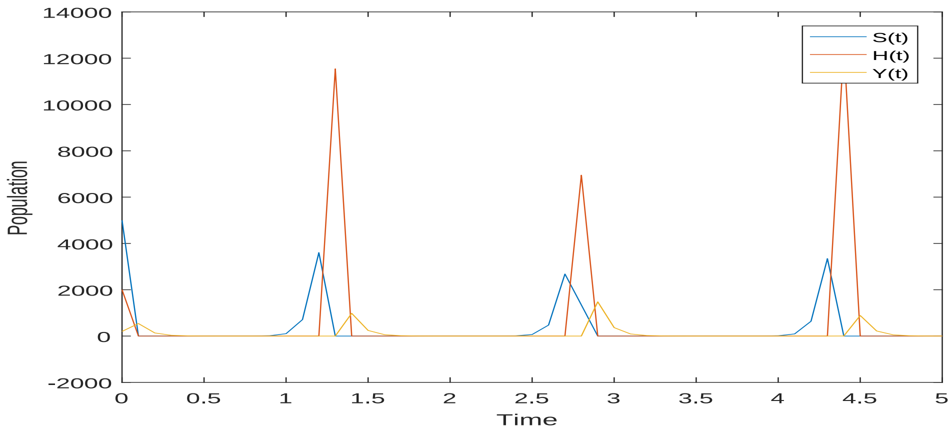

Figure 4.5: Susceptible plants, herbivores and their natural enemies interaction with $p_1 = 0.4, d = 14, \epsilon = 0.001$

Figure 4.5 shows that the three populations depend on each other. The susceptible plant population is negatively affected by harvesting activities where they are harvested at the rate ϵ and herbivores feed on them at the rate p_1. On the other hand natural enemies of herbivores depend on the availability of food (herbivores) for survival. High herbivore population negatively affects the susceptible plant population and positively affects the natural enemies of herbivore population. With the reduction of susceptible plant density, the herbivore population also reduces due to limited sources of food. Likewise, the density of the natural enemies of herbivores declines. This implies that a decrease in one species may lead to a decrease of another species and also an increase in one species density imply an increase in other species densities. The cycle occurs again and again over time.

To illustrate theorem 3.2.2, the coexistence of the susceptible plants, herbivores and natural enemies of herbivores densities is illustrated in figure 4.6

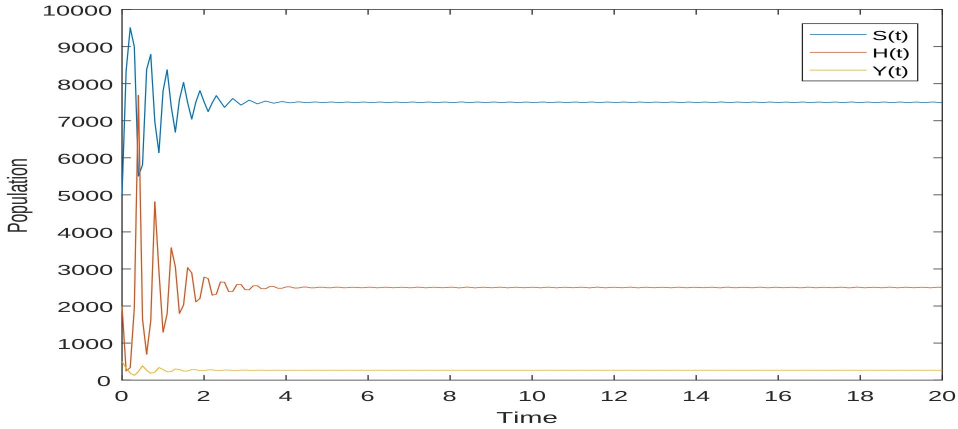

Figure 4.6: Susceptible plants, herbivores and their natural enemies interaction with $p_1 = 0.002$, d=6, $p_2 = 0.12$

From the figure 4.6, the susceptible plants, herbivores and their natural enemies population coexists without attaining the specific equilibrium at first. However in the long run, the system becomes stable and the species coexists. The coexistence of the three species demands that the initial susceptible plant population must be above the minimum required with less harvesting to sustain herbivore population. On the other hand, the average density of newborn herbivores from herbivore mother must be greater than that of natural enemies of herbivores to secure food, that is, $c_1 > c_2$. The system becomes stable where the susceptible plant density is higher followed by herbivores then their natural enemies.

4.4 Simulation of Susceptible Plants, Infected Plants, Herbivores and their Natural Enemies

For theorem 3.2.3, where there is the susceptible plants, infected plants, herbivores and their natural enemies populations. Using parameters values $r = 20$, $k = 10000$, $p_1 = 0.002$, $c_1 = 1.22$, $\theta = 0.4$, $\mu = -0.14$, $p_2 = 0.12$,

$c_2 = 0.02, \epsilon = 0.001$, $a = 10$, and $\sigma = 0.025$. The following simulations in figure 4.7, figure 4.8, and Figure 4.9 are obtained

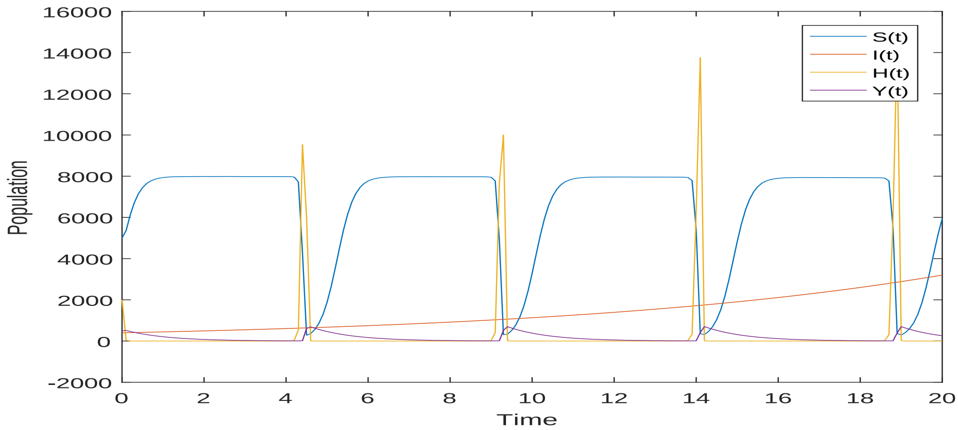

Figure 4.7: Plants, pathogen and herbivore interaction incorporating Allee effect and harvesting with $r = 4.8, d = 1, \eta = 1.3$

From figure 4.7, the susceptible plants, herbivores and their natural enemies depend on each other. Presence of pathogens, herbivores and harvesting activities negatively affects the susceptible plant population where pathogens reduces the susceptible plant density at the rate σ. When pathogens, herbivore and harvesting of plant decrease in the habitat, the susceptible plant population grows bounded by the environmental carrying capacity. In the long run, this attract herbivores who may migrate to such habitat due to availability of food. This justify why μ is negative, which shows that herbivores migrate to the habitat where there is food. This in turn attract their natural enemies in the same habitat. When herbivore density, pathogens and harvest of plants density is high, the natural enemies of herbivore increases while on other hand the susceptible plant populations decreases. Decrease of susceptible plant population result into decrease of herbivores which in turn leads to decrease of natural enemies of herbivore population. When susceptible plants reduces to certain threshold, they regenerate again and the cycle occurs again.

Moreover, the susceptible plants, infected plants, herbivores and their natural enemies population coexists without attaining a specific equilibrium point at first. However, in the long run, when consumption rate of herbivores, predation rate of natural enemies of herbivores and harvesting of susceptible plants is lower while the intrinsic growth rate of susceptible plants is higher, the system becomes stable and coexists as seen in figure 4.8.

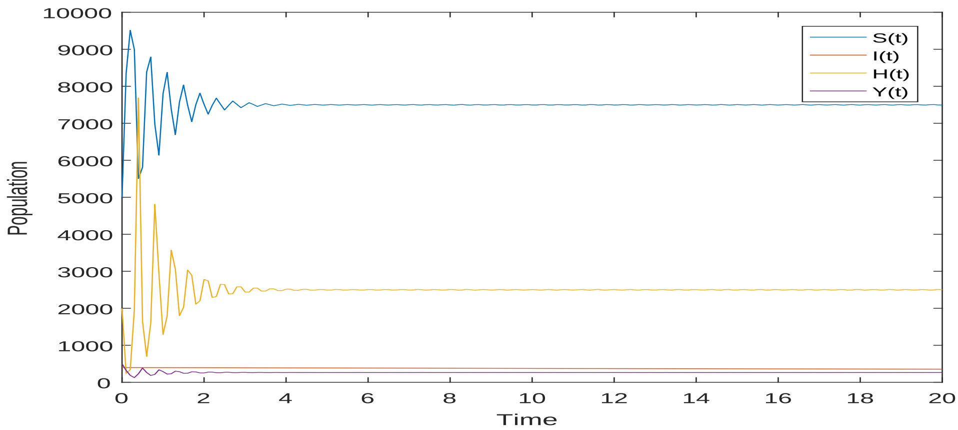

Figure 4.8: Plants, pathogen and herbivore interaction incorporating Allee effect and harvesting with $\eta = 0.2, d = 6$

Infected plants may increase or decrease depending on prevailing climatic condition in a certain habitat and measures taken by the susceptible plants to protect themselves from pathogen attack. Moreover, host plants may be resistance to pathogens or other pathogens must be at a critical stage in order to cause infections while others have evolved and therefore they can live for a prolonged periods such as brown spot. On the other hand, some susceptible plants releases VOC and HIVP to protect themselves. For this case, the species may coexist as seen in figure 4.9

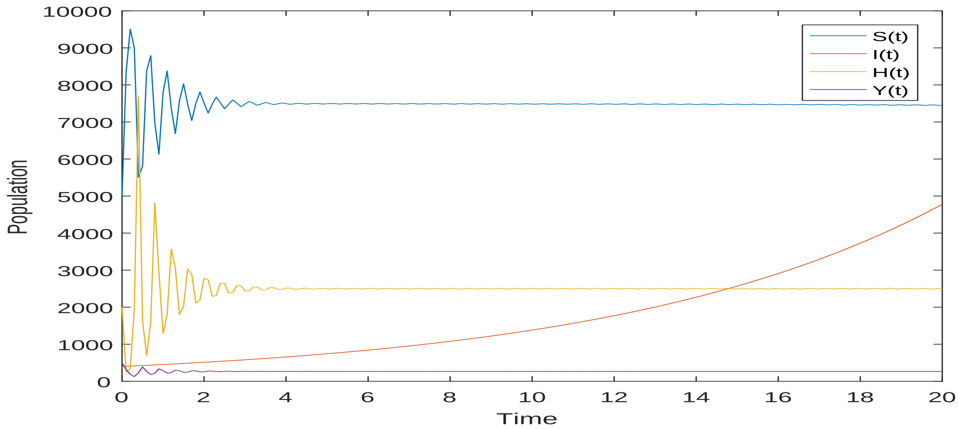

Figure 4.9: Plants, pathogen and herbivore interaction with incorporating Allee effect and harvesting with $\eta = 1.5$

Most species are prone to extinction especially herbivore population which is assumed to reproduce sexually. Allee effect plays an important role for coexistence. In absence of Allee effect, say $\theta = 0$ regardless of the other parameters, dynamics of susceptible plants, infected plants, herbivores and their natural enemies over time is simulated using parameter values in table 4.5.

Fig	r	k	p_1	c_1	θ	μ	p_2	c_1	d	ϵ	η	a	σ
Fig 4.10	3.8	10000	4	2	0	0.3	0.2	0.25	2	0.1	2	1	1.6

Table 4.4: Parameters value of susceptible plants, infected plants,herbivores incorporating harvesting in absence of Allee effect

Performing numerical simulation using values in table 4.5, we obtain figure 4.10

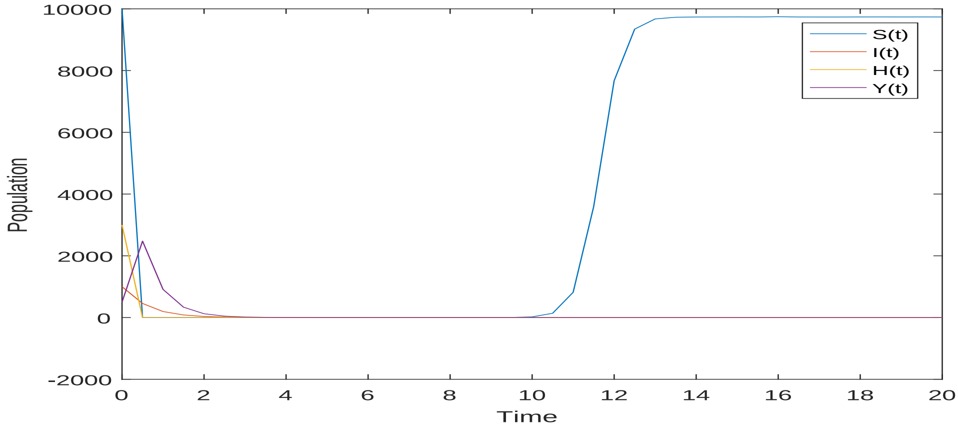

Figure 4.10: Plants-pathogens-herbivores interactions incorporating harvesting in the absence of Allee effect

From figure 4.10, when the susceptible plant population declines, herbivores start to decrease and goes to zero when the herbivore density is less than the least number of herbivores required to keep the population existing in the system. The herbivore population becomes extinct regardless of availability of susceptible plants to feed on. That is when $\theta = 0$, there will be no least number of herbivores. This leads to decrease of their natural enemies of herbivores to extinction in the same habitat. For some time, susceptible population regenerates and grows to the carrying capacity since there is less or no feeding.

Therefore for coexistence of all the species in the habitat, there is need to control harvesting rate of plant population where plants are cut down, destroyed by fire and other human activities. This can be achieved by setting harvest rate at $\epsilon = 0.001$ to secure availability of food for herbivores and pathogens. On the other hand, there is need to set lower bound on herbivore population that can result in critical population thresholds below which population crash to extinction and also $c_1 > 1$ since the herbivores are assumed to reproduce sexually.

CHAPTER 5

CONCLUSION AND RECOMMENDATION

5.1 CONCLUSION

A mathematical model of plant-pathogen-herbivore interaction incorporating Allee effect and harvest was formulated. Plant population was divided into susceptible and infected plant densities with logistic growth rate for the susceptible plant population. In this model, infected plants are plants invaded by pathogens and remains in the ecosystem until they are harvested through human activities or killed by pathogen. It is assumed that herbivores do not feed on infected plants and the species interaction and consumption are assumed to be of the same type in any habitat.

For herbivore conversion rate for new ones, a linear multiple of the functional response and Allee effect were taken into consideration since herbivore population is more prone to extinction than plants. With the fact that no species is isolated and live forever in the ecosystem, constant removal rate of natural enemies of herbivore and constant removal of herbivore from the habitat was taken into account. The herbivores can be removed through natural death, killed by human or predators or migrate from the habitat. The effects of human interference in terms of the harvest rate ϵ on susceptible plants and infected plants was also considered.

The herbivore population is more vulnerable to extinction as compared to plants since they are reproduced sexually hence inclusion of Allee effect concept. Where, when there is no least number of herbivores required to keep the population existing say $\theta = 0$, the herbivore population becomes extinct regardless of the availability of susceptible plants to feed on. On the other hand, the growth of herbivores is assumed to governed by the availability of food to feed on in any habitat. That is, when there is no susceptible plants to feed on, the herbivores goes to extinction, for instance $c_1p_1(0)(\frac{H}{\theta+H}) = 0$.

Multiple equilibrium points were obtained that is E_0, E_1, E_2, E_3, E_4 and E_5. Local stability conditions for equilibrium points were obtained in terms of system parameters where E_0 , E_1, E_2, E_3,E_4 and E_5 are locally asymptotically stable under certain conditions. The equilibrium point E_4 guarantees coexistence susceptible plants, herbivores and their natural enemies and E_5 guarantees coexistence of all species including infected plants. The stability analysis showed that the ratio of intrinsic growth rate to the environmental carrying capacity of susceptible plants must be greater than certain threshold value to raise sufficient plant biomass to sustain other species. It also shows that the intrinsic growth rate of plants must be greater than the harvesting rate of plant population for plants to get established. Given this circumstance, all species coexists.

Numerical analysis of the model was performed and showed that all the species depend on each other and coexist as seen in figure 4.6, figure 4.8 and figure 4.9. The population densities increases when there is sufficient resources whereas limited resources lead to extinction of the species in certain habitat as seen in figure 4.5 and 4.7. Excessive human activities and pathogens affects susceptible plants density and lead to decline of susceptible plant density which eventually affects other species. This implies that these populations depend on each other for food and no species in isolated. Therefore, the population densities increase when there exists abundance of their food resources while, population densities decrease when there is limited resources to sustain the populations.

It also shows that all species coexist when intrinsic growth rate of plants is greater than the harvesting rate and when conversion rate of what is eaten by herbivores to newborn ones is greater than that of their natural enemies. It also shows that in the absence of susceptible plants, herbivores migrate in search of food, while others die out. Furthermore, regardless of the availability of susceptible plants, the herbivores population goes to extinction if the herbivore population is less than the lower limit required to keep the herbivores existing in the ecosystem as seen in figure 4.10.

5.2 RECOMMENDATION

For coexistence of all species, the problems that slow down species growth whether plants, herbivores, natural enemies of herbivores, or any other resources should be addressed as these species depend on reproductive surplus in a population for maintenance. It is crucial to manage the species that are available while also evaluating the likelihood of population extinction. For example, human activities that destroy natural habitats, like building infrastructure through habitats, clearing forests for settlements, farming, burning charcoal, and poaching of herbivores and their natural enemies should be addressed. This will encourage the population growth of plants, which provide habitat and food for pathogens, herbivores and their natural enemies.

In this work all parameters, interactions and consumption are assumed to be constant and of the same type in any habitat. However, in reality the predation rates of herbivore-susceptible plants and herbivores-natural enemies and harvesting rate of plants depends on the availability of the species.

.Therefore, for future study, it can be assumed that the rate of harvesting of plants, predation rate of susceptible plants-herbivore and herbivores-natural enemies should not be constant since they may depends on the availability species.

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